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Hydrodynamics of deformable contiguous spherical shapes in an incompressible inviscid fluid

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SUMMARY

An exact solution to the two-body interaction problem is presented for the case of spherical shapes moving in an incompressible and inviscid fluid. The spheres are assumed to translate in an arbitrary manner and to undergo radial deformation (or pulsation). The problem is formulated in terms of spherical harmonics and the force experienced by the spheres is obtained by employing the Lagally theorem. The expressions for the force are given as an infinite sum of coefficients which are found by solving an infinite set of linear equations. Three main geometries are considered, namely, two spheres exterior to each other, one sphere in the interior of the other and sphere in a rectangular channel. Numerical values for the added-mass coefficients as well as for the hydrodynamic forces are found for the case of rigid sphere moving toward or parallel to a rigid wall or a free surface, and a pulsating sphere in the proximity of these boundaries. Also given are numerical values for the transverse and the longitudinal added-mass coefficients for a sphere moving in a rectangular channel for different channel-blockage ratios.

1. Introduction

A common hydrodynamical problem involves the determination of the interactive forces between adjacent moving particles immersed in a fluid medium. This interaction is important for example in studies of two phase flows, bubble dynamics, mechanics of suspensions, multi-phase heat and mass transfer, combustion of droplets, settling of rain drops and hydrodynamics of biological systems. On a large scale the problem arises in the field of transportation such as the motion of bodies (or vehicles) past each other, critical collision-avoidance maneuvers as well as maneuvers in a confined or dense medium. When the distance between the particles is relatively large, their interactive force may be neglected. However, when the distance decreases so that, for example the concentration of the particles in a two-phase flow situation is of the order of one percent, the particle interaction must be considered in the analysis of the hydrodynamical problem [1]. In many cases shapes are encountered which, to a first-order approximation, may be replaced by equivalent spheres having the same volume as long as interactive effects are considered. Such spherical shapes may be rigid, undergo continuous deformation or may exhibit pulsating motion. For moderate and high Reynolds number flows, a first step towards the solution of the viscous and possibly thermal problem might be the potential flow solution. In particular the assumption of potential flow past bubble surfaces was found to be fairly accurate [2]. Potential theory has also direct applications in the analysis of superfluids such as liquid helium or the interior of a neutron star. In the hydrodynamical analysis of critical maneuvers of contiguous vehicles, a common practice is to calculate the mutual interactive force by assuming the fluid to be inviscid and incompressible. Even with such an assumption, which was found to yield practical results, the computation of the two-body interaction is a formidable task. For this reason simple geometries are usually used to approximate the shapes. For example, a two-sphere model was used in the analysis of the two-train problem [3].

Several solutions for ideal flow about two rigid spheres are available. The first classical solutions are due to Stokes [4], Basset [5], Hicks [6] and Herman [7], which are based on the method of successive images and provide approximate solutions for both the velocity potential and the kinetic energy of the fluid. Michael [8] solved the problem of potential flow past a row of identical spheres by using an electromagnetic analogy and more recently Small and Weihs [9] and Love [10] presented an exact series solution for the potential function in the two-sphere problem by employing bispherical coordinates. The sphere-sphere interactive force was considered first by Endo [11], by calculating the pressure distribution on the spheres, then by Kawaguti [3] for two spheres in collinear motion and later by Voinov [12] who calculated the force between spheres a small distance apart. The method of successive images has been also employed by Helfinstine and Dalton [13] who derived approximate expressions for the interactive force between a group of rigid stationary spheres in potential flow. To the best knowledge of the author the force between two deformable spheres in a general motion has not yet been considered.

In the present analysis a generalization of the Lagally theorem [14] and formulation in terms of spherical coordinates are used to derive exact analytic expressions for the interactive force between two moving contiguous deformable spheres. The use of spherical rather than bispherical coordinate system was found to be more convenient for the following reasons; the use of bispherical coordinates is limited to two spheres in collinear motion whereas the spherical coordinate system can handle several spheres in general motion. The bispherical system can not be applied to radial motion of spheres or to the limiting case of touching spheres. Indeed Weihs and Small [15] have shown that a tangent-sphere rather than bispherical coordinate system should be used in the limit of two touching spheres. The present solution is free of these restrictions. In addition the solution presented in [9] is based on the application of the collocation method which may lead to considerable inaccuracies due to an improper choice of the mesh points. Spherical coordinates remove the necessity of collocation and finally enable one to use the recent generalization of the Lagally theorem [14] to compute the forces acting on the spheres. Finally the present solution is expressed in terms of simple arithmetic expressions as compared with the complicated algebraic functions used in [9] and [15].

First an expression for the force acting on a single sphere moving in an arbitrary unsteady potential flow is derived. The general result thus obtained is used to compute the interactive force between two spheres in general motion. In addition to the force, expressions for the kinetic energy and for the added-mass coefficients are also derived.

Three different geometries are considered, namely, two spheres moving in an infinite medium bounded internally by the two spheres, one sphere moving in the interior of another sphere, and a single sphere moving in a two dimensional rectangular channel. The first, the "exterior" case, is related to the common two-body problem, the second "interior" case has direct application to multiphase heat and mass studies employing the shell-model [16] and the third case is encountered in the body-in-a-channel or the classical "blockage" problem.

2. The Lagally force on a single sphere

A single sphere moving through an unsteady and non-uniform inviscid incompressible flow field will experience a hydrodynamical force. In a Cartesian coordinate system attached to the center of the sphere let $\boldsymbol{R}(x, y, z)$ denote the radius vector and $V_c(U_c, V_c, W_c)$ the absolute velocity of the sphere. In addition the sphere is allowed to have a uniform time-dependent radial velocity $\dot{a}$ where a is the instantaneous radius of the sphere. The image system (inside the sphere) of the external flow field may be considered as a series of multipoles lying at the center of the sphere. The exterior potential field may thus be given by

$$\phi'(x, y, x) = -\sum_s (-1)^n M_n \frac{\partial^n}{\partial x^\alpha \partial y^\beta \partial z^\gamma}\left(\frac{1}{R}\right), \qquad R^2 = x^2 + y^2 + z^2, \tag{1}$$

where α, β, γ and n are integers such that $n = \alpha + \beta + \gamma$, M_n is the strength of a general multipole of order n and $\sum_s$ denotes summation over all particular values of α, β, and γ representing the multipoles distribution.

The generalized Lagally theorem [14], applied to a deformed spherical shape, yields the following expression for the hydrodynamical force acting on the sphere.

$$\boldsymbol{F} = \rho\left\{\frac{\partial}{\partial t}\left[V_c \mathcal{V} - 4\pi \sum_s M_n \frac{\partial^n}{\partial x^\alpha \partial y^\beta \partial z^\gamma}(\boldsymbol{R})_0\right] - 4\pi \sum_s M_n \frac{\partial^n}{\partial x^\alpha \partial y^\beta \partial z^\gamma}(\boldsymbol{v})_0\right\}, \tag{2}$$

where ρ is the fluid density, $\mathcal{V}$ denotes the volume of the sphere and $\boldsymbol{v}$ is the velocity induced at the location of the multipoles by all external flow producing mechanisms. The subscript $_0$ denotes that the various partial derivatives of $\boldsymbol{R}$ and $\boldsymbol{v}$ should be evaluated at the origin, and t denotes time.

Let the total velocity potential in the flow domain exterior to the sphere be given by the following series of spherical harmonics:

$$\begin{aligned}\phi(x, y, z) &= a[U\phi_1(x, y, z) + V\phi_2(x, y, z) + W\phi_3(x, y, z) + \dot{a}\phi_0(x, y, z)] \\ &= a\{\dot{a} \sum_{n=0}^{\infty} [D_n(R/a)^n + \tilde{D}_n(a/R)^{n+1}]P_n(\mu) + U \sum_{n=1}^{\infty} [A_n(R/a)^n + \tilde{A}_n(a/R)^{n+1}]P_n(\mu) \\ &\quad + V \sum_{n=1}^{\infty} [B_n(R/a)^n + \tilde{B}_n(a/R)^{n+1}]P_n^1(\mu)\cos\psi \\ &\quad + W \sum_{n=1}^{\infty} [C_n(R/a)^n + \tilde{C}_n(a/R)^{n+1}]P_n^1(\mu)\sin\psi\}\end{aligned} \tag{3}$$

where (U, V, W) denote the three components of the characteristic translatory velocity vector V and $\dot{a}$ is the radial velocity of the sphere. Note that for a moving sphere $V_c = V$ and for a stationary sphere $V_c = 0$. The Legendre polynomials are denoted by $P_n^m(\mu)$ where $\mu = \cos\theta$ such that the transformation between the Cartesian (x, y, z) and the spherical (R, θ, ψ) coordinate systems is given by

$$x = R\mu, \quad y = R(1-\mu^2)^{\frac{1}{2}}\cos\psi, \quad z = R(1-\mu^2)^{\frac{1}{2}}\sin\psi. \tag{4}$$

The eight dimensionless coefficients in (3) namely, A_n, B_n, C_n, D_n, and $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{D}_n$, are to be found from the following boundary conditions on the surface of the sphere:

$$\frac{\partial \phi}{\partial R} = \dot{a} + U_c \mu + V_c(1-\mu^2)^{\frac{1}{2}} \cos \psi + W_c(1-\mu^2)^{\frac{1}{2}} \sin \psi \tag{5}$$

which yield

$$\begin{Bmatrix} \tilde{A}_n \\ \tilde{B}_n \\ \tilde{C}_n \end{Bmatrix} = -\tfrac{1}{2}\delta(n-1) \begin{Bmatrix} U_c/U \\ V_c/V \\ W_c/W \end{Bmatrix} + \frac{n}{n+1} \begin{Bmatrix} A_n \\ B_n \\ C_n \end{Bmatrix}, \qquad n = 1, 2, 3, \ldots, \tag{6}$$

and

$$\tilde{D}_n = -\delta(n) + \frac{n}{n+1} D_n, \qquad n = 0, 1, 2, \ldots \tag{7}$$

Here $\delta(m)$ is the Kronecker delta function which is one for $m = 0$ and zero otherwise.

By applying the general formula [17],

$$\frac{\partial^{n-m}}{\partial x^{n-m}} \left(\frac{\partial}{\partial y} + i \frac{\partial}{\partial x} \right)^m \left\{ \frac{1}{R} \right\} = (-1)^n (n-m)!\, R^{-(n+1)} P_n^m(\mu) e^{im\psi}, \tag{8}$$

it can be shown that the image of the exterior potential field (1) is given by the following distribution of multipoles at the origin:

$$\phi'(x, y, z) = -\dot{a}a^2/R + a^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} a^n \frac{\partial^{n-1}}{\partial x^{n-1}}$$
$$\times \left[(U\tilde{A}_n + \dot{a}\tilde{D}_n) \frac{\partial}{\partial x} + nV\tilde{B}_n \frac{\partial}{\partial y} + nW\tilde{C}_n \frac{\partial}{\partial z} \right] \left\{ \frac{1}{R} \right\} \tag{9}$$

where ϕ' denotes the vanishing-at-infinity part of the velocity potential given in (3). The remaining part of the velocity potential (non-singular at the origin) will contribute to the non-uniform velocity induced at the origin by all external flow-producing mechanisms, hence,

$$\boldsymbol{v}(x, y, z) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \phi''(x, y, z) \tag{10}$$

where ϕ'' is the non-uniform vanishing-at-origin part of the velocity potential given by

$$\phi''(x, y, z) = a \sum_{n=2}^{\infty} [(UA_n + \dot{a}D_n)P_n(\mu) + (VB_n \cos \psi + WC_n \sin \psi) P_n^1(\mu)](R/a)^n. \tag{11}$$

At this stage it is convenient to refer to a useful relation [17], namely

$$\frac{\partial^k}{\partial x^k} \{R^n P_n^m(\mu) e^{im\psi}\} = \frac{(n+m)!}{(n+m-k)!} R^{n-k} P_{n-k}^m(\mu) e^{im\psi} \tag{12}$$

which is valid for $n - m \geq k$ and gives zero on the right hand side of (12) for $n - m < k$.

Equation (12) can be used to derive the following expressions:

$$\frac{\partial^m}{\partial x^{m-k}\partial y^k}\{R^n P_n(\mu)\}_0 = \frac{\partial^m}{\partial x^{m-k}\partial z^k}\{R^n P_n(\mu)\}_0 = m![\delta(k) - \tfrac{1}{2}\delta(k-2)], \tag{13}$$

$$\frac{\partial^m}{\partial x^{m-k}\partial y^k}\{R^n P_n^1(\mu)\cos\psi\}_0$$
$$= \frac{\partial^m}{\partial x^{m-k}\partial z^k}\{R^n P_n^1(\mu)\sin\psi\}_0 = \tfrac{1}{2}(m+1)!\,\delta(k-1), \tag{14}$$

which are valid for $k = 0, 1, 2$ and $(m, n) > 0$. The subscript 0 in (13) and (14) again denotes that the partial derivatives have to be evaluated at the origin.

Substituting eqs. (6), (7), (13) and (14) into the Lagally theorem (2) yields the following expressions for the three components of the hydrodynamical force $\boldsymbol{F}(F_x, F_y, F_z)$:

$$F_x = \tfrac{4}{3}\pi\rho\frac{\partial}{\partial t}[a^3(U_c + 3U\tilde{A}_1)]$$
$$+ 2\pi\rho a^2 \sum_{n=0}^{\infty}(n+2)\{2[U^2\tilde{A}_n\tilde{A}_{n+1} + U\dot{a}(\tilde{A}_n\tilde{D}_{n+1} + \tilde{D}_n\tilde{A}_{n+1})$$
$$+ \dot{a}^2\tilde{D}_n\tilde{D}_{n+1}] + n(n+2)[V^2\tilde{B}_n\tilde{B}_{n+1} + W^2\tilde{C}_n\tilde{C}_{n+1}]\}, \tag{15a}$$

$$F_y = \tfrac{4}{3}\pi\rho\frac{\partial}{\partial t}[a^3(V_c + 3V\tilde{B}_1)] + 2\pi a^2\rho\sum_{n=0}^{\infty}(n+2)\{UV[(n+2)\tilde{A}_n\tilde{B}_{n+1} - n\tilde{B}_n\tilde{A}_{n+1}]$$
$$+ \dot{a}V[(n+2)\tilde{D}_n\tilde{B}_{n+1} - n\tilde{B}_n\tilde{D}_{n+1}]\}, \tag{15b}$$

$$F_z = \tfrac{4}{3}\pi\rho\frac{\partial}{\partial t}[a^3(W_c + 3W\tilde{C}_1)] + 2\pi a^2\rho\sum_{n=0}^{\infty}(n+2)\{UW[(n+2)\tilde{A}_n\tilde{C}_{n+1} - n\tilde{C}_n\tilde{A}_{n+1}]$$
$$+ \dot{a}W[(n+2)\tilde{D}_n\tilde{C}_{n+1} - n\tilde{C}_n\tilde{D}_{n+1}]\}. \tag{15c}$$

The above expressions are symmetric with respect to y and z but not with respect to x because of the particular form chosen for the exterior flow field in equation (3). When the sphere moves in an otherwise undisturbed medium the summations in eqs. (15) vanish and the classical result for the force experienced by a sphere in unsteady motion is obtained. It will be also shown later that $1 + 3\hat{A}_1$, $1 + 3\hat{B}_1$ and $1 + 3\hat{C}_1$ denote the negative values of the three added-mass coefficients of the sphere in the x, y and z-directions respectively.

As a demonstration of the use of eqs. (15), the force experienced by a stationary rigid sphere in a steady "constant shear" flow with velocity potential,

$$\phi''(x, y, z) = a\{U[x/a + \bar{\alpha}(2x^2 - y^2 - z^2)/a^2]$$
$$+ V[y/a + \bar{\beta}xy/a^2] + W[z/a + \bar{\gamma}xz/a^2]\} \tag{16}$$

is calculated where $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ are some prescribed coefficients. Substitution of (16) into (11) and using (6) yields $\tilde{A}_1 = \tilde{B}_1 = \tilde{C}_1 = \frac{1}{2}$, $\tilde{A}_2 = 4\bar{\alpha}/3$, $\tilde{B}_2 = 2\bar{\beta}/3$ and $\tilde{C}_2 = 2\bar{\gamma}/3$. The rest of the coefficients in (9) are zero. Inserting these values into (15) renders

$$F_x = 2\pi\rho a^2(4\bar{\alpha}U^2 + 3\bar{\beta}V^2 + 3\bar{\gamma}W^2), \tag{17a}$$

$$F_y = 2\pi\rho a^2 UV(3\bar{\beta} - 2\bar{\alpha}), \tag{17b}$$

$$F_z = 2\pi\rho a^2 UW(3\bar{\gamma} - 2\bar{\alpha}). \tag{17c}$$

The general expressions for the hydrodynamical force acting on a single sphere moving in a disturbed medium will be used to compute the force on a moving sphere due to the proximity of an adjacent sphere. In what follows the general motion of two contiguous spheres is analysed. The general velocity potential is expressed as a sum of four Kirchhoff potentials in the manner described in equation (3). Here ϕ_1 denotes the unit velocity potential due to motion of the spheres along the line joining their centers (taken to be the x-axis) where ϕ_2 and ϕ_3 are the unit velocity potentials corresponding to motions in the y- and z-directions respectively in the plane normal to the line of centers. Finally ϕ_0 denotes the unit potential due to the radial motion of the stationary spheres. In this manner series solution for the four coefficients $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$ and $\tilde{D}_n$ are obtained which enable the interactive force to be calculated directly from equations (15).

3. External motion of two rigid spheres along line of centers

Consider the axisymmetric flow resulting from the motion of two contiguous rigid spheres in the direction of the line joining their centers. The distance between the centers of the two spheres, of radii a_1 and a_2, is denoted by b as depicted in Fig. 1. The origin of the Cartesian coordinate system is chosen to be at the center of the sphere a_1 such that the x-axis is aligned along the line of centers. In such a coordinate system the centers of spheres a_1 and a_2 are at (0, 0, 0) and (b, 0, 0) and their translatory velocities along the x-axis are denoted by U_1 and U_2 respectively. In addition, two spherical coordinate systems (R_1, θ_1, ψ_1) and (R_2, θ_2, ψ_2), which are attached to the centers of spheres a_1 and a_2 respectively, are defined by

$$\begin{aligned} x &= R_1 \cos\theta_1 = R_2 \cos\theta_2 + b, \\ y &= R_1 \sin\theta_1 \cos\psi_1 = R_2 \sin\theta_2 \cos\psi_2, \\ z &= R_1 \sin\theta_1 \sin\psi_1 = R_2 \sin\theta_2 \sin\psi_2. \end{aligned} \tag{18}$$

The far field behaviour of the velocity potential in the domain exterior to the spheres is given by

$$\phi_1 = U_1 a_1 \sum_{n=1}^{\infty} \tilde{A}_n^1 P_n(\mu_1)(a_1/R_1)^{n+1} + U_2 a_2 \sum_{n=1}^{\infty} \tilde{A}_n^2 P_n(\mu_2)(a_2/R_2)^{n+1}, \tag{19}$$

where $\tilde{A}_n^1$ and $\tilde{A}_n^2$ are coefficients to be determined and $\mu_1 = \cos\theta_1$, $\mu_2 = \cos\theta_2$.

On the assumption of rigid spheres, a Neumann type boundary condition, applied on the surface of the spheres, implies

$$\frac{\partial\phi_1}{\partial R_1} = U_1\mu_1 \qquad \text{on } R_1 = a_1, \tag{20a}$$

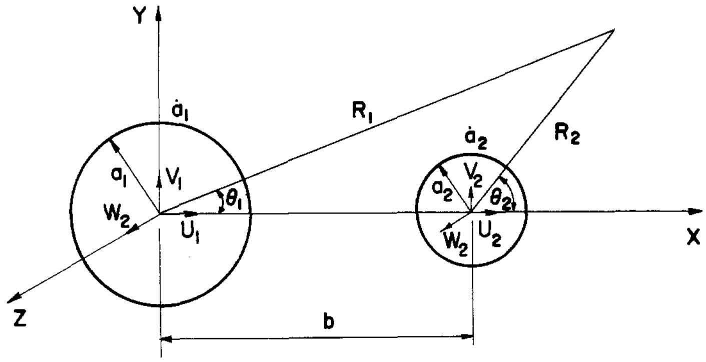

Figure 1. The "external" case.

$$\frac{\partial \phi_1}{\partial R_2} = U_2 \mu_2 \qquad \text{on } R_2 = a_2. \tag{20b}$$

To apply these boundary conditions to (19), use is made of some transformations [17],

$$P_n^m(\mu_2) R_2^{-(n+1)} = \sum_{k=0}^{\infty} (-1)^{n-m} \frac{(n+m+k)!}{(n-m)!(2m+k)!} b^{-(n+m+k+1)} P_{m+k}^m(\mu_1) R_1^{m+k}, \tag{21}$$

$$P_n^m(\mu_1) R_1^{-(n+1)} = \sum_{k=0}^{\infty} (-1)^{k} \frac{(n+m+k)!}{(n-m)!(2m+k)!} b^{-(n+m+k+1)} P_{m+k}^m(\mu_2) R_2^{m+k}, \tag{22}$$

where equation (21) is valid for $R_1 < b$ and (22) for $R_2 < b$.

Substituting (21) into (19) yields for the latter

$$\phi_1(R_1, \mu_1) = U_1 a_1 \sum_{n=1}^{\infty} \tilde{A}_n^1 P_n(\mu_1)(a_1/R_1)^{n+1}$$
$$+ U_2 a_2 \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \tilde{A}_m^2 (-1)^m \frac{(n+m)!}{n!m!} \lambda_2^{m+1} \lambda_1^n P_n(\mu_1)(R_1/a_1)^n, \tag{23}$$

where $\lambda_1 = a_1/b$ and $\lambda_2 = a_2/b$. Applying (20a) to (23) together with the orthogonality properties of the Legendre polynomials, yield the following relation between $\tilde{A}_n^1$ and $\tilde{A}_n^2$:

$$\tilde{A}_n^1 = -\tfrac{1}{2}\delta(n-1) + \frac{n\varepsilon_1}{(n+1)!} \lambda_1^{n-1} \sum_{m=1}^{\infty} \tilde{A}_m^2 (-1)^m \frac{(m+n)!}{m!} \lambda_2^{m+2}, \tag{24}$$

where $\varepsilon_1 = U_2/U_1$. Repeating the same procedure of substituting (22) into (19) and using (20b) yields

$$\tilde{A}_n^2 = -\tfrac{1}{2}\delta(n-1) + \frac{n}{\varepsilon_1(n+1)!} (-1)^n \lambda_2^{n-1} \sum_{m=1}^{\infty} \tilde{A}_m^1 \frac{(m+n)!}{m!} \lambda_1^{m+2}. \tag{25}$$

The two sets of linear equations (24) and (25) can be solved for the coefficients $\tilde{A}_n^1$, yielding

$$\sum_{k=1}^{\infty} \tilde{A}_k^1 Y_n^k = \chi_n, \tag{26}$$

where

$$\chi_n = \tfrac{1}{2}\delta(n-1) + \tfrac{1}{2}n\varepsilon_1\lambda_2^3\lambda_1^{n-1} \tag{27a}$$

and

$$Y_n^k = -\delta(n-k) + \frac{n}{k!(n+1)!}\lambda_1^{n+k+1}\sum_{m=1}^{\infty}\frac{(m+n)!(m+k)!}{(m-1)!(m+1)!}\lambda_2^{2m+1}. \tag{27b}$$

Once the $\tilde{A}_n^1$ are found, the coefficients $\tilde{A}_n^2$ may be obtained from (25). In the case where one of the spheres is stationary, say $U_j = 0$ $(j = 1, 2)$, the above equations are still valid providing $U_j\tilde{A}_n^j$ is finite. It may be noted that for the particular case of identical spheres, $a_1 = a_2$ and $\varepsilon_1 = \pm 1$, the solution of (24) and (25) implies that

$$\tilde{A}_n^2 = (-1)^{n+1}\tilde{A}_n^1. \tag{28}$$

For the case of identical spheres the coefficients may be found directly from

$$\tilde{A}_n^1 = -\tfrac{1}{2}\delta(n-1) - \frac{n\varepsilon_1}{(n+1)!}\sum_{m=1}^{\infty}\tilde{A}_m^1\frac{(m+n)!}{m!}\lambda_1^{m+n+1}. \tag{29}$$

A quantity of interest is also the kinetic energy T_1 of the fluid exterior to the spheres which is given by

$$-T_1 = \tfrac{1}{2}\rho U_1\int_{S_1}\phi_1(R_1,\mu_1)\mu_1\,dS_1 + \tfrac{1}{2}\rho U_2\int_{S_2}\phi_1(R_2,\mu_2)\mu_2\,dS_2, \tag{30}$$

where S_1 and S_2 denote the surface of the spheres a_1 and a_2 respectively. Substituting (19) into (30) and performing the integration renders a rather simple result for the kinetic energy of the fluid, namely,

$$-T_1 = \tfrac{2}{3}\pi\rho[a_1^3U_1^2(1+3\tilde{A}_1^1) + a_2^3U_2^2(1+3\tilde{A}_1^2)] \tag{31}$$

implying also that $-(1+3\tilde{A}_1^1)$ and $-(1+3\tilde{A}_1^2)$ are the longitudinal added-mass coefficients of spheres a_1 and a_2, respectively, for translation along the x-axis.

4. External motion of two rigid spheres normal to line of centers

Here the interaction between two spheres moving in a plane which is normal to the line joining their centers is analysed. First the case where the two spheres are translating with velocities V_1 and V_2 in the y-direction is solved. Because of symmetry the results thus obtained are also applicable for the case where the spheres are moving in the z-direction with velocities W_1 and W_2.

In the same coordinate system used in the previous section, the exterior velocity potential resulting from the motion of the spheres in the y-direction may be written in the following form:

$$\phi_2 = V_1 a_1 \sum_{n=1}^{\infty} \tilde{B}_n^1 P_n^1(\mu_1)(a_1/R_1)^{n+1} \cos\psi_1 + V_2 a_2 \sum_{n=1}^{\infty} \tilde{B}_n^2 P_n^1(\mu_2)(a_2/R_2)^{n+1} \cos\psi_2, \tag{32}$$

where $\psi_1 = \psi_2$, and both $\tilde{B}_n^1$ and $\tilde{B}_n^2$ are to be found. The proper boundary conditions on the surfaces of the spheres are

$$\frac{\partial\phi_2}{\partial R_1} = V_1 P_1^1(\mu_1)\cos\psi_1 \qquad \text{on } R_1 = a_1, \tag{33a}$$

and

$$\frac{\partial\phi_2}{\partial R_2} = V_2 P_1^1(\mu_2)\cos\psi_2 \qquad \text{on } R_2 = a_2. \tag{33b}$$

Using the transformation (21), equation (32) may be rewritten in the following two equivalent forms:

$$\phi_2(R_1,\mu_1,\psi_1) = V_1 a_1 \sum_{n=1}^{\infty} \tilde{B}_n^1 P_n^1(\mu_1)(a_1/R_1)^{n+1}\cos\psi_1$$
$$- V_2 a_2 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \tilde{B}_m^2(-1)^m \frac{(m+n)!}{(m-1)!\,(n+1)!}\lambda_2^{m+1}\lambda_1^n P_n^1(\mu_1)(R_1/a_1)^n\cos\psi_1, \tag{34}$$

or

$$\phi_2(R_2,\mu_2,\psi_2) = V_2 a_2 \sum_{n=1}^{\infty} \tilde{B}_n^2 P_n^1(\mu_2)(a_2/R_2)^{n+1}\cos\psi_2$$
$$- V_1 a_1 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \tilde{B}_m^1(-1)^n \frac{(m+n)!}{(m-1)!\,(n+1)!}\lambda_1^{m+1}\lambda_2^n P_n^1(\mu_2)(R_2/a_2)^n\cos\psi_2, \tag{35}$$

which together with (33a) and (33b) yield

$$\tilde{B}_n^1 = -\tfrac{1}{2}\delta(n-1) - \frac{n\varepsilon_2}{(n+1)(n+1)!}\lambda_1^{n-1}\sum_{m=1}^{\infty}\tilde{B}_m^2(-1)^m\frac{(m+n)!}{(m-1)!}\lambda_2^{m+2}, \tag{36}$$

and

$$\tilde{B}_n^2 = -\tfrac{1}{2}\delta(n-1) - \frac{n}{\varepsilon_2(n+1)(n+1)!}(-1)^n\lambda_2^{n-1}\sum_{m=1}^{\infty}\tilde{B}_m^1\frac{(m+n)!}{(m-1)!}\lambda_1^{m+2} \tag{37}$$

where $\varepsilon_2 = V_2/V_1$. Solving (36) and (37) for $\tilde{B}_n^1$ gives the following set of linear equations:

$$\sum_{k=1}^{\infty}\tilde{B}_k^1 Y_n^k = \chi_n, \tag{38}$$

where

$$\chi_n = \tfrac{1}{2}\delta(n-1) + \tfrac{1}{2}\frac{n\varepsilon_2}{n+1}\lambda_1^{n-1}\lambda_2^3 \tag{39}$$

and

$$Y_n^k = -\delta(n-k) + \frac{n}{(n+1)(n+1)!\,(k-1)!}\lambda_1^{n+k+1}\sum_{m=1}^{\infty}\frac{m(m+n)!\,(m+k)!}{(m+1)(m-1)!\,(m+1)!}. \tag{40}$$

Having found $\tilde{B}_n^1$, one obtains the coefficients $\tilde{B}_n^2$ from (37). When one of the spheres is stationary, say $V_j = 0$ $(j = 1, 2)$, the above analysis holds for $V_j\tilde{B}_n^j$ finite. For the particular case of identical spheres, $a_1 = a_2$ and $\varepsilon_2 = \pm 1$,

$$\tilde{B}_n^2 = (-1)^{n+1}\tilde{B}_n^1, \tag{41}$$

and the coefficients $\tilde{B}_n^1$ may be found directly from the simplified version of (36) namely,

$$\tilde{B}_n^1 = -\tfrac{1}{2}\delta(n-1) + \frac{n\varepsilon_2}{(n+1)(n+1)!}\sum_{m=1}^{\infty}\tilde{B}_m^1\frac{(m+n)!}{(m-1)!}\lambda_1^{m+n+1}. \tag{42}$$

The kinetic energy of the exterior fluid, T_2, is given by

$$\begin{aligned} -T_2 &= \tfrac{1}{2}\rho V_1\int_{S_1}\phi_2(R_1,\mu_1,\psi_1)P_1^1(\mu_1)\cos\psi_1\,dS_1 \\ &\quad + \tfrac{1}{2}\rho V_2\int_{S_2}\phi_2(R_2,\mu_2,\psi_2)P_1^1(\mu_2)\cos\psi_2\,dS_2, \end{aligned} \tag{43}$$

yielding

$$-T_2 = \tfrac{2}{3}\pi\rho[a_1^3V_1^2(1+3\tilde{B}_1^1) + a_2^3V_2^2(1+3\tilde{B}_1^2)] \tag{44}$$

again, implying that the longitudinal added-mass coefficients for translation along the y-axis, are $-(1+3\tilde{B}_1^1)$ and $-(1+3\tilde{B}_1^2)$ for sphere a_1 and a_2 respectively. All the expressions derived in this section are also valid for the coefficients $\tilde{C}_n^1$ and $\tilde{C}_n^2$ corresponding to motion in the z-direction, provided $\tilde{B}_n^1$, $\tilde{B}_n^2$ are replaced by $\tilde{C}_n^1$, $\tilde{C}_n^2$, respectively, and ε_2 is replaced by $\varepsilon_3 = W_2/W_1$.

5. External motion due to two adjacent pulsating spheres

Here the spheres are assumed to pulsate with radial velocities $\dot{a}_1$ and $\dot{a}_2$ while the centers of the spheres remain fixed in a space. Let the velocity potential in the exterior domain be given by

$$\phi_0 = \dot{a}_1a_1\sum_{n=0}^{\infty}\tilde{D}_n^1P_n(\mu_1)(a_1/R_1)^{n+1} + \dot{a}_2a_2\sum_{n=0}^{\infty}\tilde{D}_n^2P_n(\mu_2)(a_2/R_2)^{n+1}. \tag{45}$$

$\tilde{D}_n^1$ and $\tilde{D}_n^2$ are some unknown coefficients, and the boundary conditions on the spheres are

$$\frac{\partial \phi_0}{\partial R_1} = \dot{a}_1 \text{ on } R_1 = a_1; \qquad \frac{\partial \phi_0}{\partial R_2} = \dot{a}_2 \text{ on } R_2 = a_2. \tag{46}$$

Using the transformations given in (21) and (22), one may express ϕ_0 as

$$\phi_0(R_1, \mu_1) = \dot{a}_1 a_1 \sum_{n=0}^{\infty} \tilde{D}_n^1 P_n(\mu_1)(a_1/R_1)^{n+1}$$
$$+ \dot{a}_2 a_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \tilde{D}_m^2 (-1)^m \frac{(m+n)!}{m!\,n!} \lambda_2^{m+1} \lambda_1^n P_n(\mu_1)(R_1/a_1)^n, \tag{47}$$

or

$$\phi_0(R_2, \mu_2) = \dot{a}_2 a_2 \sum_{n=0}^{\infty} \tilde{D}_n^2 P_n(\mu_2)(a_2/R_2)^{n+1}$$
$$+ \dot{a}_1 a_1 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \tilde{D}_m^1 (-1)^n \frac{(n+m)!}{n!\,m!} \lambda_1^{m+1} \lambda_2^n P_n(\mu_2)(R_2/a_2)^n, \tag{48}$$

which together with (46) yields the following two sets of linear equations:

$$\tilde{D}_n^1 = -\delta(n) + \frac{n\varepsilon_0}{(n+1)!} \lambda_1^{n-1} \sum_{m=0}^{\infty} \tilde{D}_m^2 (-1)^m \frac{(m+n)!}{m!} \lambda_2^{m+2} \tag{49}$$

and

$$\tilde{D}_n^2 = -\delta(n) + \frac{n}{\varepsilon_0 (n+1)!} (-1)^n \lambda_2^{n-1} \sum_{m=0}^{\infty} \tilde{D}_m^1 \frac{(m+n)!}{m!} \lambda_1^{m+2}, \tag{50}$$

where $\varepsilon_0 = \dot{a}_2/\dot{a}_1$ (implying also that $\tilde{D}_0^1 = \tilde{D}_0^2 = -1$). Solving (49) and (50) for $\tilde{D}_n^1$ gives

$$\sum_{k=0}^{\infty} \tilde{D}_k^1 Y_n^k = \chi_n, \tag{51}$$

where

$$\chi_n = \delta(n) + \frac{n\varepsilon_0}{n+1} \lambda_1^{n-1} \lambda_2^2 \tag{52}$$

and

$$Y_n^k = -\delta(n-k) + \frac{n}{k!\,(n+1)!} \lambda_1^{n+k+1} \sum_{m=0}^{\infty} \frac{(m+n)!\,(m+k)!}{(m-1)!\,(m+1)!} \lambda_2^{2m+1}. \tag{53}$$

For the case where $a_1 = a_2$ and $\varepsilon_0 = \pm 1$ we have $\tilde{D}_n^1 = (-1)^n \tilde{D}_n^2$, where $\tilde{D}_n^1$ may be found from a simplified version of (49),

$$\tilde{D}_n^1 = -\delta(n) + \frac{n\varepsilon_0}{(n+1)!} \sum_{m=0}^{\infty} \tilde{D}_m^1 \frac{(m+n)!}{m!} \lambda_1^{n+m+1}. \tag{54}$$

Finally the kinetic energy of the fluid excited by the pulsation of the spheres, T_0, is given by

$$-T_0 = \tfrac{1}{2}\rho\dot{a}_1 \int_{S_1} \phi_0(R_1, \mu_1)dS_1 + \tfrac{1}{2}\rho\dot{a}_2 \int_{S_2} \phi_0(R_2, \mu_2)dS_2. \quad (55)$$

Substituting (47) and (48) into (55), one obtains after performing the integration

$$-T_0 = 2\pi\rho\{a_1^3\dot{a}_1^2[-1 + \varepsilon_0\lambda_1^{-1} \sum_{m=0}^{\infty} \tilde{D}_m^2(-1)^m\lambda_2^{m+2}]$$

$$+ a_2^3\dot{a}_2^2[-1 + \varepsilon_0^{-1}\lambda_2^{-1} \sum_{m=0}^{\infty} \tilde{D}_m^1\lambda_1^{m+2}]\}, \quad (56)$$

which cannot, like the previous cases, be expressed merely in terms of a single coefficient.

6. Internal motion along line of centers of two deforming spheres

Here the motion of sphere a_1 with velocity U_1, in the interior of a stationary sphere a_2 (Fig. 2) is considered. Both spheres are assumed to have radial velocities $\dot{a}_1$ and $\dot{a}_2$ which must satisfy the relation

$$\dot{a}_1 a_1^2 = \dot{a}_2 a_2^2, \quad (57)$$

since the fluid bounded by the two spheres is incompressible. The total velocity potential in the interior fluid domain is expressed as

$$\phi(R_1, \mu_1) = U_1 a_1 \sum_{n=0}^{\infty} [E_n(R_1/a_1)^n + \tilde{E}_n(a_1/R_1)^{n+1}]P_n(\mu_1), \quad (58)$$

where (R_1, μ_1) and (R_2, μ_2) are axisymmetric spherical coordinate systems attached to the center of spheres a_1 and a_2 respectively, and both E_n and $\tilde{E}_n$ are prescribed coefficients. The

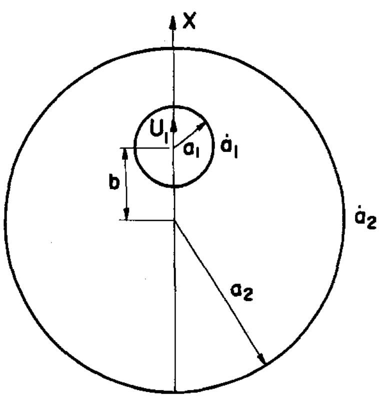

Figure 2. The "internal" case.

boundary conditions on the spherical surfaces are

$$\frac{\partial\phi}{\partial R_1} = U_1\mu_1 + \dot{a}_1 \qquad \text{on } R_1 = a_1, \tag{59a}$$

and

$$\frac{\partial\phi}{\partial R_2} = \dot{a}_2 \qquad \text{on } R_2 = a_2. \tag{59b}$$

Applying (59a) to (58) yields

$$\tilde{E}_n = -\frac{\dot{a}_1}{U_1}\delta(n) - \tfrac{1}{2}\delta(n-1) + \frac{n}{n+1}E_n. \tag{60}$$

To transform (58) into the (R_2, μ_2) coordinate system the following transformations are used [17]:

$$P_n(\mu_1)(a_1/R_1)^{n+1} = \sum_{m=0}^{\infty} \frac{(-1)^m(m+m)!}{n!m!} \lambda_1^{n+1}\lambda_2^{-(n+m+1)}P_{n+m}(\mu_2)(a_2/R_2)^{n+m+1} \tag{61}$$

and

$$P_n(\mu_1)(R_1/a_1)^n = \sum_{m=0}^{\infty} \frac{n!}{m!(n-m)!}\lambda_2^{n-m}\lambda_1^{-n}P_{n-m}(\mu_2)(R_2/a_2)^{n-m}, \tag{62}$$

by which equation (58) may be expressed as

$$\begin{aligned}\phi(R_2, \mu_2) = {} & U_1 a_1 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E_{n+m}\frac{(n+m)!}{n!m!}\lambda_2^n\lambda_1^{-(n+m)}P_n(\mu_2)(R_2/a_2)^n \\ & + U_1 a_1 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \tilde{E}_{n-m}\frac{(-1)^m n!}{m!(n-m)!}\lambda_2^{-(n+1)}\lambda_1^{n-m+1}P_n(\mu_2)(a_2/R_2)^{n+1}.\end{aligned} \tag{63}$$

Applying (59b) to (63) gives

$$\begin{aligned}&\sum_{m=0}^{\infty} E_{n+m}\frac{n(n+m)!}{m!(n+1)!}\lambda_1^{-m} - \sum_{m=0}^{\infty}\tilde{E}_{n-m}\frac{(-1)^m n!}{m!(n-m)!}\lambda_1^{-m}\lambda_2^{-(2n+1)} \\ &\quad = (\lambda_2/\lambda_1)(\dot{a}_2/U_1)\delta(n).\end{aligned} \tag{64}$$

Solving equations (60) and (64) for E_n gives

$$\sum_{k=0}^{\infty} E_k Y_n^k = \chi_n, \tag{65}$$

where

$$\chi_n = (\lambda_2/\lambda_1)(\dot{a}_2/U_1)\delta(n) - (\dot{a}_1/U_1)(-1)^n\lambda_1^{-n}\lambda_2^{-(2n+1)} + \tfrac{1}{2}n(-1)^n\lambda_1^{1-n}\lambda_2^{-(2n+1)} \quad (66)$$

and

$$Y_n^k = \frac{nk!}{(n+1)!(k-n)!}H(k-n) - (-1)^{n-k}\frac{n}{n+1}\frac{n!}{k!(n-k)!}\lambda_1^{k-n}\lambda_2^{-(2n+1)}H(n-k), \quad (67)$$

where $H(k)$ is the Heaviside step function which is unity for $k \geq 0$ and zero for $k < 0$.

7. Sphere moving in a two-dimensional rectangular channel

Consider the motion of a sphere of radius a in a two dimensional rectangular channel of width b at the instance the sphere's center lies on the center-line of the channel as depicted in Fig. 3. In a Cartesian coordinate system with the origin at the center of the sphere, the channel plane walls are given by $x = \pm b/2$. The three components of the translatory velocity are denoted by (U, V, W) and $\dot{a}$ is the sphere radial velocity. It is also assumed for simplicity that the fluid in the channel is disturbed only by the motion of the sphere. In this example, which obviously can not be solved by employing bispherical coordinates, it is of interest to determine the force and the added mass coefficients of the sphere.

To account for the Neumann boundary conditions on the channel walls, an infinite row of image-spheres with centers at $(jb, 0, 0)$ is considered. Here j is an integer such that $-\infty < j < \infty$. In addition, spherical coordinate systems are defined so that the system (R_j, μ_j, ψ_j) lies at the center of the j^{th} sphere namely at $(jb, 0, 0)$. The value of $j = 0$ corresponds to the original coordinate system such that $(R, \mu, \psi) = (R_0, \mu_0, \psi_0)$ and similarly $\psi_j = \psi$ for all values of j.

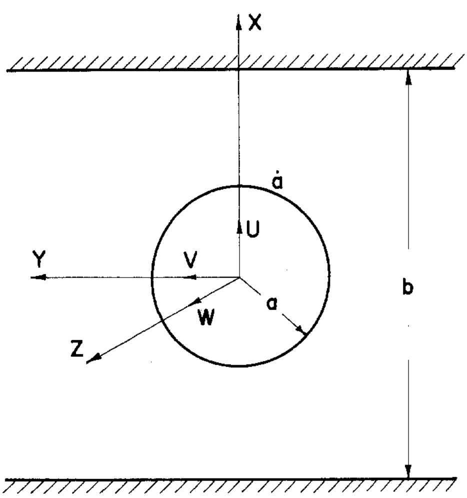

Figure 3. Sphere in a channel.

Using these notations, one can express the velocity potential in the channel as follows:

$$\phi(R,\mu,\psi) = Ua\phi_1(R,\mu,\psi) + Va\phi_2(R,\mu,\psi) + Wa\phi_3(R,\mu,\psi) + \dot{a}a\phi_0(R,\mu,\psi)$$

$$= Ua \sum_{j=-\infty}^{\infty} \sum_{n=1}^{\infty} \tilde{A}_n(-1)^{|j|} P_n(\mu_j)(a/R_j)^{n+1} + Va \sum_{j=-\infty}^{\infty} \sum_{n=1}^{\infty} \tilde{B}_n P_n^1(\mu_j)(a/R_j)^{n+1} \cos\psi$$

$$+ Wa \sum_{j=-\infty}^{\infty} \sum_{n=1}^{\infty} \tilde{C}_n P_n^1(\mu_j)(a/R_j)^{n+1} \sin\psi + \dot{a}a \sum_{j=-\infty}^{\infty} \sum_{n=0}^{\infty} \tilde{D}_n P_n(\mu_j)(a/R_j)^{n+1} \qquad (68)$$

The velocity potential on the left-hand side of (68) automatically satisfies the boundary conditions on the channel walls, namely $\partial\phi/\partial x = 0$ at $x = \pm b/2$. The unknown coefficients in (68) are to be determined from the following boundary conditions on the sphere:

$$\frac{\partial\phi_1}{\partial R} = P_1(\mu)/a, \qquad \frac{\partial\phi_2}{\partial R} = P_1^1(\mu)\cos\psi/a, \qquad \frac{\partial\phi_3}{\partial R} = P_1^1(\mu)\sin\psi/a, \qquad (69)$$

$$\frac{\partial\phi_0}{\partial R} = P_0(\mu)/a \qquad \text{on } R = a.$$

The equation for the determination of the coefficients $\tilde{A}_n$ will be first obtained. Using the two transformations given in (21) and (22), the velocity potential ϕ_1, defined in (68) may be also written as

$$\phi_1(R,\mu,\psi) = \sum_{n=1}^{\infty} \tilde{A}_n P_n(\mu)(a/R)^{n+1} + \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \tilde{A}_m(-1)^j[(-1)^n + (-1)^m]$$

$$\times \frac{(m+n)!}{m!n!} (\lambda/j)^{n+m+1} P_n(\mu)(R/a)^n. \qquad (70)$$

The infinite sum over all positive integers j in (70) may be expressed in terms of the Riemann zeta function $\zeta(n)$ defined by

$$\zeta(n) = \sum_{j=1}^{\infty} j^{-n}, \qquad (71)$$

since it can be shown that

$$\sum_{j=1}^{\infty} (-1)^j j^{-n} = (2^{1-n} - 1)\zeta(n). \qquad (72)$$

Substituting (72) into (70) gives

$$\phi_1(R,\mu,\psi) = \sum_{n=1}^{\infty} \tilde{A}_n P_n(\mu)(a/R)^{n+1} - \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \tilde{A}_m[(-1)^m + (-1)^n][1 - 2^{-(n+m)}]$$

$$\times \zeta(n+m+1) \frac{(n+m)!}{n!m!} \lambda^{n+m+1} P_n(\mu)(R/a)^n. \qquad (73)$$

Applying the boundary condition at $R = a$, (69), results in the following expression for $\tilde{A}_n$:

$$\tilde{A}_{2n+1} = -\tfrac{1}{2}\delta(n) + \frac{2(2n+1)}{(2n+2)!}\sum_{m=0}^{\infty}\tilde{A}_{2m+1}[1 - 2^{-(2n+2m+2)}]$$
$$\times\, \zeta(2n+2m+3)\frac{(2n+2m+2)!}{(2m+1)!}\lambda^{2n+2m+3}, \tag{74}$$

and $\tilde{A}_{2n} = 0$.

Following the same procedure, one can show that the coefficients $\tilde{B}_n = \tilde{C}_n$ are to be found from

$$(\tilde{B}, \tilde{C})_{2n+1} = -\tfrac{1}{2}\delta(n) + \frac{2n+1}{(n+1)(2n+2)!}$$
$$\times \sum_{m=0}^{\infty}(\tilde{B}, \tilde{C})_{2m+1}\zeta(2n+2m+3)\frac{(2n+2m+2)!}{(2m)!}\lambda^{2n+2m+3} \tag{75}$$

and $\tilde{B}_{2n} = 0$.

Finally $\tilde{D}_n$ are given by the solution of the following series:

$$\tilde{D}_{2n} = -\delta(n) + \frac{4n}{(2n+1)!}\sum_{m=0}^{\infty}\tilde{D}_{2m}\zeta(2n+2m+1)\frac{(2n+2m)!}{(2m)!}\lambda^{2n+2m+1}, \tag{76}$$

where $\tilde{D}_{2n+1} = 0$.

Applying equations (15) to (68) and recalling that $\tilde{A}_{2n} = \tilde{B}_{2n} = \tilde{C}_{2n} = \tilde{D}_{2n+1} = 0$ yields the following expressions for the three components of the hydrodynamical force experienced by the sphere in the channel:

$$F_x = \tfrac{4}{3}\pi\rho\frac{\partial}{\partial t}[a^3(U_c + 3U\tilde{A}_1)] + 4\pi\rho U\dot{a}a^2\sum_{n=0}^{\infty}(2n+3)\tilde{A}_{2n+1}\tilde{D}_{2n+2}, \tag{77a}$$

$$F_y = \tfrac{4}{3}\pi\rho\frac{\partial}{\partial t}[a^3(V_c + 3V\tilde{B}_1)] - 2\pi\rho V\dot{a}a^2\sum_{n=0}^{\infty}(2n+1)(2n+3)\tilde{B}_{2n+1}\tilde{D}_{2n+2}, \tag{77b}$$

$$F_z = \tfrac{4}{3}\pi\rho\frac{\partial}{\partial t}[a^3(W_c + 3W\tilde{C}_1)] - 2\pi\rho W\dot{a}a^2\sum_{n=0}^{\infty}(2n+1)(2n+3)\tilde{C}_{2n+1}\tilde{D}_{2n+2}, \tag{77c}$$

where $(U_c, V_c, W_c) = (U, V, W)$ for a moving sphere in a quiescent medium.

It is thus shown that a rigid sphere in a general steady translatory motion exhibits no force at the instant its center lies on the centerline of the channel. It is only the combination of the radial and the translatory motions of the sphere that yield a force.

8. Numerical examples and discussions

As an illustration of the general analysis for the two-sphere problem (Fig. 1), we consider the case of two identical spheres, $a_1 = a_2$. Of particular interest is the case where the two spheres

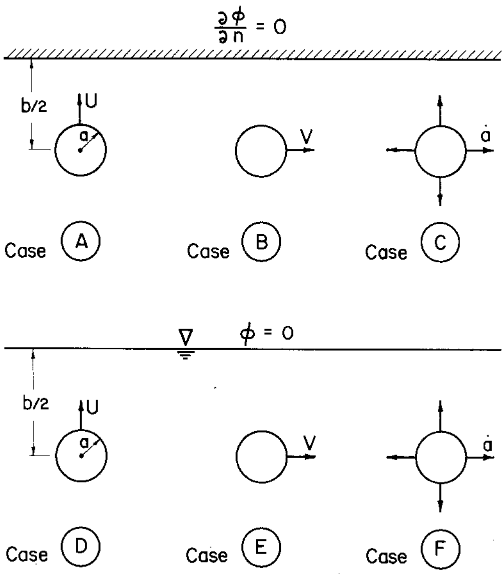

Figure 4. Notation for the six cases corresponding to the motion of a sphere near a rigid wall or a free surface.

move in an arbitrary manner (including deformation) such that the absolute values of the velocity components of both spheres are equal, namely, $\varepsilon_j = \pm 1$ for $j = 0, 1, 2, 3$. With a proper choice of the directions of motion of the two spheres, the plane of symmetry (normal to the line of centers) can be made an impermeable surface (Neumann problem) or alternatively an equipotential surface (Dirichlet problem). The first case corresponds to the motion of a sphere near a plane wall, whereas the second case represents the motion of a submerged sphere beneath a free surface when the Froude number is large (infinite). When the two spheres move in opposite directions, the plane of symmetry is impermeable and by considering the half-space problem, the three cases depicted in Fig. 4, namely translatory motion of a sphere toward (A) a rigid wall, pulsating (C) near a rigid wall and translation parallel to a free surface (E), are obtained. Similarly, when the two spheres move in the same direction, the plane of symmetry is equipotential, and cases B, D and F depicted in Fig. 4, corresponding to pulsating and translatory motions toward a free surface, as well as translation parallel to a rigid wall, are obtained.

For these conditions, the coefficients $\tilde{A}_n^1$, $\tilde{B}_n^1(\tilde{C}_n^1)$ and $\tilde{D}_n^1$ may be found directly from equations (29), (42) and (54) respectively. These infinite sets of linear equations were solved by the method of reduction [19]. In this method, the solution is found by solving a sequence of finite systems, each of which is obtained from the infinite set by discarding all equations and unknowns beyond a certain number N. The value of N was chosen so as to yield a maximum relative error of $\pm 10^{-4}$ between successive approximations. This test was applied only to the case where the convergence was the slowest, namely the case of touching spheres ($b/a = 2$) and the value of $N = 30$ was found to fit this error criterion. The various coefficients computed for $N = 30$ and ($b/a = 2$) are given in Table 1. Only the value of $\tilde{A}_1^1$ for touching spheres and $\varepsilon_1 = -1$ can be checked against an exact solution available in the literature [9, 12]. Since the added-mass coefficient for a translation along the lines of centers

TABLE 1

The coefficients $\tilde{A}_n^1$, $\tilde{B}_n^1$ and $\tilde{D}_n^1$ for two identical spheres in contact where $N = 30$, (only the first fifteen values are given)

n	$-\tilde{A}_n^1$		$-\tilde{B}_n^1$		$\tilde{D}_n^1$	
	$\varepsilon_1 = -1$	$\varepsilon_1 = 1$	$\varepsilon_2 = -1$	$\varepsilon_2 = 1$	$\varepsilon_0 = -1$	$\varepsilon_0 = 1$
0	0.0	0.0	0.0	0.0	1.0	1.0
1	.600638	.450771	.473265	.540324	−.104381	.177142
2	.118952	−.045490	−.016768	.029989	−.059422	.155655
3	.110335	−.030605	−.008666	.019522	−.026169	.123798
4	.095491	−.017495	−.004100	.012580	−.009159	.099638
5	.081314	−.008701	−.001792	.008301	−.001753	.081924
6	.069525	−.003596	−.000701	.005675	.000914	.069290
7	.060110	−.000966	−.000220	.004032	.001506	.059808
8	.052610	.000204	−.000027	.002970	.001323	.052416
9	.046553	.000599	.000037	.002257	.000942	.046460
10	.041560	.000628	.000048	.001761	.000585	.041529
11	.037360	.000514	.000041	.001403	.000317	.037358
12	.033759	.000368	.000029	.001137	.000141	.033767
13	.030623	.000237	.000019	.000934	.000037	.030632
14	.027854	.000136	.000011	.000776	−.000018	.027861
15	.025382	.000066	.000006	.000651	−.000042	.025386

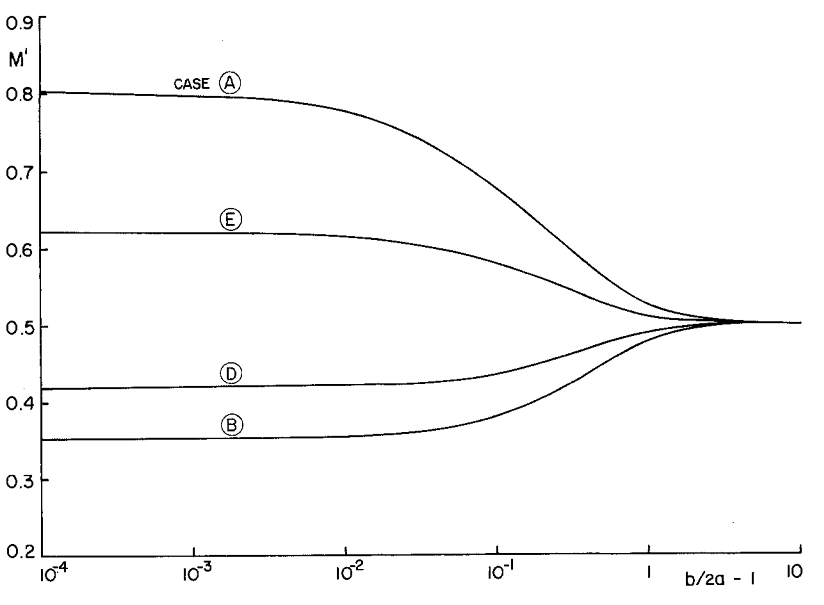

Figure 5. Added mass coefficients for cases A, B, D and E depicted in Fig. 4.

is $|1 + 3\tilde{A}_1^1|$, it can be shown that $\tilde{A}_1^1 = -\frac{1}{2}\zeta(3) = -0.601028$ where $\zeta(n)$ is the Riemann zeta function, implying that the limiting added-mass coefficient is $M' = 0.803084$. It can also be shown that in the limit of touching spheres and $\varepsilon_1 = +1$, one gets $\tilde{A}_1^1 = -\frac{3}{8}\zeta(3) = -0.450771$, and the added-mass coefficient is $M' = 0.352313$. For values of $b/a > 2$, the convergence of the iteration procedure was faster and the values of the coefficients were found to decrease rapidly with a further increase in b. Curves depicting the variation of the added-mass coefficients with the spacing b for cases A, B, D and E are given in Fig. 5. The added-mass coefficients are simply given by $|1 + 3\tilde{A}_1^1|$ for a normal (to the boundary) motion and by $|1 + 3\tilde{B}_1^1|$ for a parallel (to the boundary) motion. The limiting values of the added-mass coefficients (correct to within four significant figures) for the touching case, where found to be 0.8020, 0.3523, 0.4198 and 0.6210 for cases A, B, D and E respectively. The added-mass coefficient of two stationary spheres in contact in a uniform flow, has been also reported in [13], where the values of 0.35 and 0.61 for cases B and E, respectively, were found numerically.

The added-mass coefficient for a motion of a sphere toward or parallel to a rigid wall can be also compared with the corresponding values for the case of a sphere moving in a two-dimensional rectangular channel (Fig. 3). The variation of the longitudinal (y-direction) and the transverse (x-direction) added-mass coefficients with the spacing between the walls is depicted in Fig. 6. The transverse added-mass coefficient is given by $-M' = 1 + 3\tilde{A}_1$ and the lontitudinal one by $-M' = 1 + 3\tilde{B}_1$. Here the coefficients $\tilde{A}_n$ and $\tilde{B}_n$ are found by solving (74) and (75) respectively. It is interesting to note that in the limiting case where the sphere is in contact with the walls $\tilde{A}_1 = -.693091$ and $\tilde{B}_1 = -.601242$ implying that the "transverse" and "longitudinal" added-mass coefficients are 1.079273 and .803726 respectively.

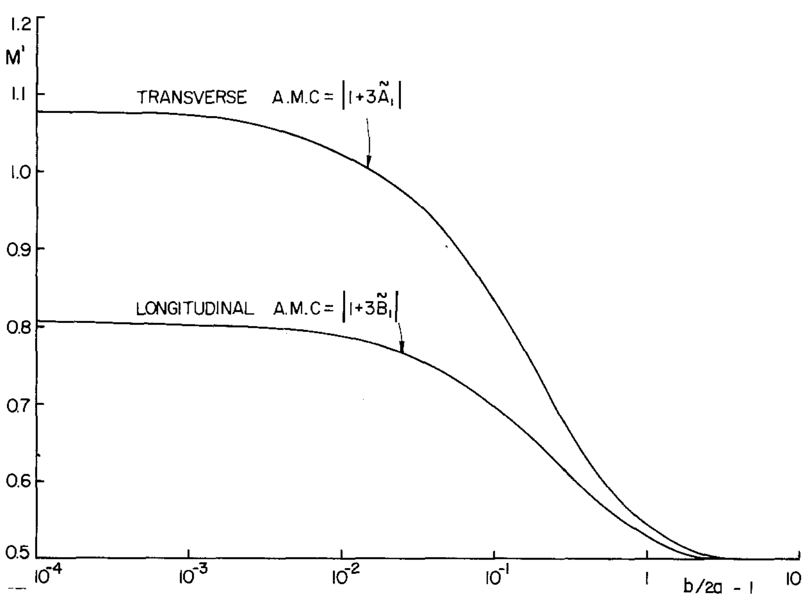

Figure 6. The longitudinal (y-direction) and transverse (x-direction) added mass coefficients for a sphere in a channel as depicted in Fig. 3.

In each one of the six modes of motion depicted in Figure 4 the sphere will experience a hydrodynamical force which depends on the spacing parameter b. The only component of the force that is not identically zero is the x-component which tends to attract (or repel) the sphere from the wall (or free surface). For the case of a uniform velocity (no time dependency), equation (15a) yields the following expression for the force experienced by the sphere in cases A and D as defined in Fig. 4:

$$F_x = 4\pi\rho a^2 \left\{ U \frac{\partial \tilde{A}_1^1}{\partial t} + U^2 \sum_{n=0}^{\infty} (n+2) \tilde{A}_n^1 \tilde{A}_{n+1}^1 \right\}, \tag{78}$$

where $\partial \tilde{A}_1^1/\partial t = 0$ in case D.

Similarly, from (15a) the hydrodynamical force for cases B and E is given by

$$F_x = 2\pi\rho a^2 V^2 \sum_{n=0}^{\infty} n(n+2)^2 \tilde{B}_n^1 \tilde{B}_{n+1}^1, \tag{79}$$

and for cases C and F by

$$F_x = 4\pi\rho a^2 \dot{a}^2 \sum_{n=0}^{\infty} (n+2) \tilde{D}_n^1 \tilde{D}_{n+1}^1. \tag{80}$$

To find the value of $\partial \tilde{A}_1^1/\partial t$ in (78) for the case of a sphere approaching a wall (case A), we differentiate equation (29) with respect to time which gives the following equation:

$$\frac{\partial \tilde{A}_n^1}{\partial t} = -\frac{2Un}{b(n+1)!} \sum_{m=1}^{\infty} \tilde{A}_m^1 \frac{(m+n+1)!}{m!} \lambda_1^{m+n+1} + \frac{n}{(n+1)!} \sum_{m=1}^{\infty} \frac{\partial \tilde{A}_m^1}{\partial t} \cdot \frac{(m+n)!}{m!} \lambda_1^{m+n+1}. \tag{81}$$

The coefficients $\tilde{A}_m^1$ have been found by solving (29), and the same numerical procedure may be also used to solve (81) since it can be shown that

$$\frac{\partial \tilde{A}_1^1}{\partial t} = -2(U/a) \sum_{n=0}^{\infty} (n+2) \tilde{A}_n^1 \tilde{A}_{n+1}^1. \tag{82}$$

To prove (82) we note that the kinetic energy of the fluid bounded by the rigid wall and the sphere may be written as $T = \frac{1}{2}\bar{M}' U^2$ where $\bar{M}'$ is the added-mass, $\bar{M}' = -\frac{4}{3}\pi\rho a^3(1 + 3\tilde{A}_1^1)$, and U is the velocity of the sphere. If we denote the instantaneous distance of the sphere from the wall by b', Lagrange's equation ([4], p. 190) yields the following expression for the force on the sphere:

$$F_x = -\frac{d}{dt}(\bar{M}'U) - \tfrac{1}{2}U^2 \frac{d\bar{M}'}{db'} = \tfrac{1}{2}U^2 \frac{d\bar{M}'}{db'}, \tag{83}$$

since $db'/dt = -U$. Comparing (83) with (78) renders the following relations:

$$-\frac{d}{dt}(\bar{M}'U) = 4\pi\rho U a^2 \frac{\partial \tilde{A}_1^1}{\partial t} = -4\pi\rho a^2 U^2 \frac{\partial \tilde{A}_1^1}{\partial b'}, \tag{84}$$

$$-\tfrac{1}{2}U^2 \frac{d\bar{M}'}{db'} = 2\pi\rho a^3 U^2 \frac{\partial \tilde{A}_1^1}{\partial b'} = 4\pi\rho a^2 U^2 \sum_{n=0}^{\infty} (n+2)\tilde{A}_n^1 \tilde{A}_{n+1}^1, \tag{85}$$

by which (82) is obtained.

Substituting (82) in (78), we get the following expression for the hydrodynamical force experienced by a sphere moving toward (or away) from a rigid wall;

$$F_x = -4\pi\rho a^2 U^2 \sum_{n=0}^{\infty} (n+2)\tilde{A}_n^1 \tilde{A}_{n+1}^1. \tag{86}$$

The coefficients $\tilde{A}_n^1$ ($\varepsilon_1 = -1$) and $\tilde{B}_n^1$ ($\varepsilon_2 = 1$) found previously were substituted in (79) and (86) yielding the numerical values for the forces experienced by a sphere moving toward or parallel to a rigid wall. The variation of these forces with the spacing parameter $-b/2$ (distance from the wall) is depicted in Figure 7. The exact solution thus found was also compared with the available classical approximate ([18], p. 538) solution for the force acting on a sphere moving with uniform velocity U toward a wall:

$$F_x \approx 6\pi\rho a^2 U^2 (a/b)^4, \tag{87}$$

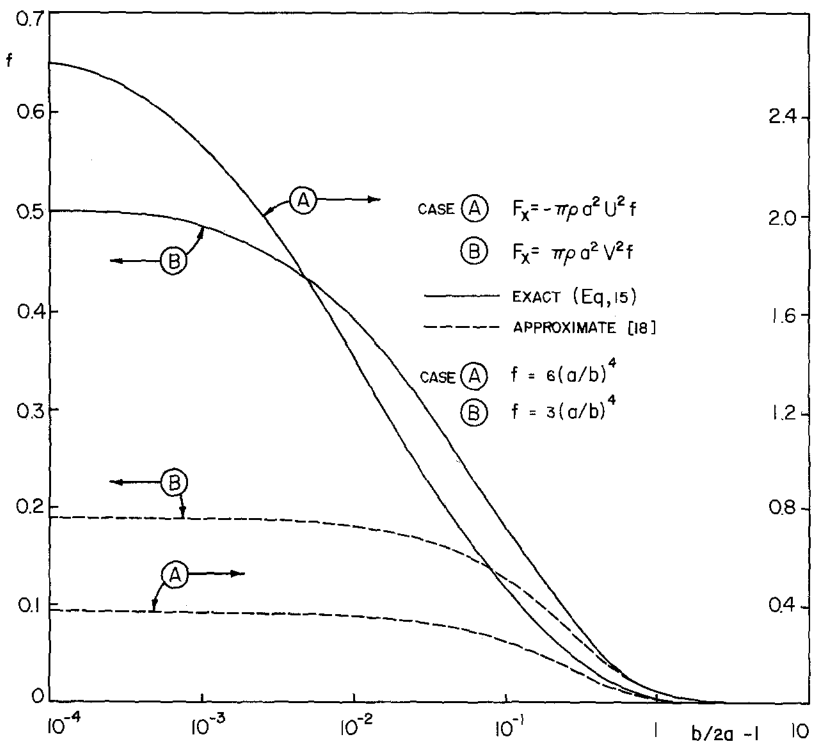

Figure 7. Exact versus approximate [18] solution for the force on a sphere moving toward (case A) and parallel (case B) to a rigid wall.

and for a sphere moving with uniform velocity V parallel to a rigid wall, namely,

$$F_x \approx 3\pi\rho a^2 V^2 (a/b)^4. \tag{88}$$

Figure 7 demonstrates the limitation of the classical solution, based on the method of successive images, for small spacing between the sphere and the wall. As an example for $b/(2a) = 1.1$, the error between the exact and the approximate solutions shown in Figure 7 is larger than 100% for case A and about 50% for case B. This error increases considerably with a further decrease in the spacing b.

Figure 8 depicts the variation of the hydrodynamical force with the spacing b for a sphere moving toward (case D) and parallel (case E) to a free surface as obtained from (78) and (79) for $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$.

For the limiting case D when the spheres touch, Voinov [12] found the following exact solution for the interaction force by employing Lagrange's equation:

$$F_x \approx \pi\rho a^2 U^2[\tfrac{3}{4}\zeta(3) - \log 2] = 0.2084\,\pi\rho a^2 U^2. \tag{91}$$

This solution can also be compared with the present solution by substituting the coefficients $\tilde{A}_n^1$ for $\varepsilon_1 = 1$ (listed in Table 1) in (78) which yields

$$F_x \approx 0.208395\,\pi\rho a^2 U^2 \tag{92}$$

in agreement with Voinov's solution (91).

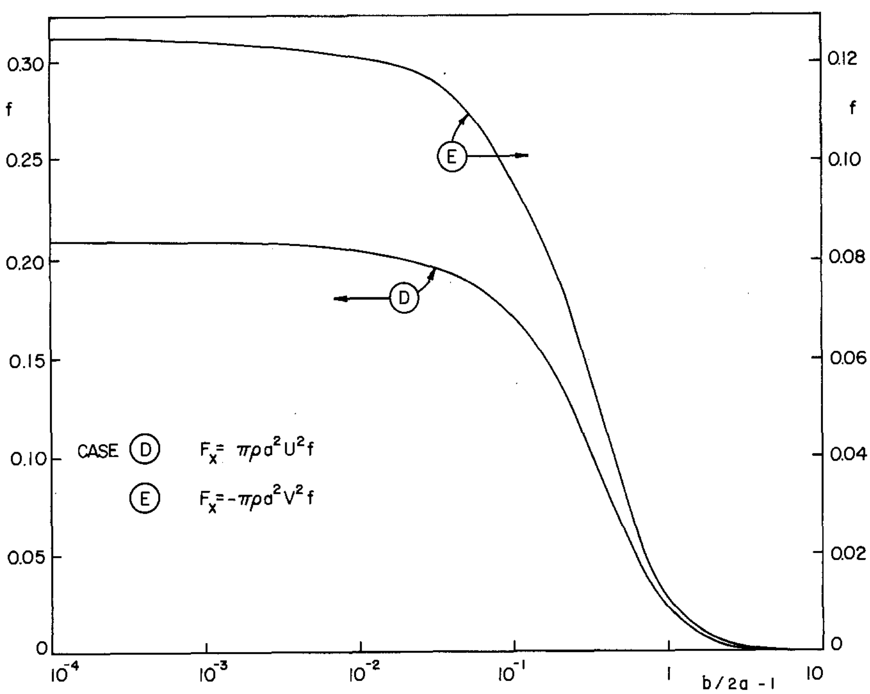

Figure 8. Variation of the force with the distance from a free surface for a sphere moving toward (case D) and parallel (case E) to it.

The force acting on a deforming sphere with a constant rate of radial deformation, in the proximity of a rigid or free surface, is depicted in Figure 9. The exact solution found by substituting the previously found coefficients $\tilde{D}_n^1$ in (80) may be also compared with the approximate solution suggested by Leahy [20]. Leahy's solution simply suggests that the force experienced by the pulsating sphere varies with the inverse of the distance from the boundary. Figure 9 demonstrates that the agreement between the approximate and the exact solutions is considerably better for a sphere pulsating near a free surface (case F) than for a sphere near a rigid wall (case C).

Finally, it should be noted that according to our sign convention a positive value of the force implies that the sphere is attracted toward the wall whereas negative values imply a repulsive force. Hence Figures 7–9 show that the sphere is attracted to the wall in the following three cases: translation parallel to a rigid wall, translation toward a free surface and pulsation near a free surface (cases B, D and F). Similarly, the sphere is repelled from the wall when it moves toward a rigid wall, moves parallel to a free surface and pulsates near a rigid wall (cases A, C and E). These results may also be obtained using the steady Bernoulli equation by considering the velocity and the pressure on the side of the sphere next to the wall and on the further side.

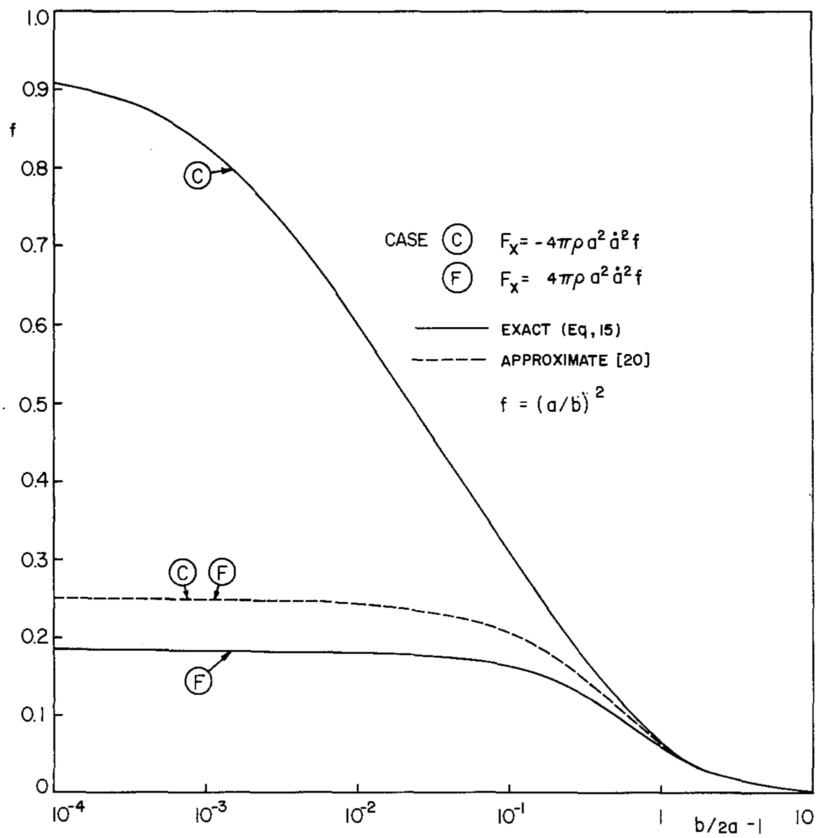

Figure 9. Exact versus approximate solution for a deforming sphere in the proximity of a rigid wall (case C) and a free surface (case F).

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